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WAVES IN A SEMI-INFINITE PLATE IN SMOOTH CONTACT WITH A HARMONICALLY DISTURBED HALF-SPACE

L. B. FREUND* and J. D. ACHENBACH

Department of Civil Engineering, Northwestern University
Evanston, Illinois

Abstract—Wave motion of a semi-infinite elastic plate in smooth contact with an elastic half-space is studied. The waves in the plate are generated by a plane time-harmonic dilatational wave incident on the surface of the half-space. By means of Laplace transform methods and the Wiener–Hopf technique an analytical expression for the plate deflection is obtained. The deflection includes terms associated with body waves in the half-space, with surface waves peculiar to a plate–half-space interface and with the geometrical reflection of the incident wave. The deflections associated with an interface wave and with the geometrical reflection of the incident wave have constant amplitudes along the entire length of the plate. At large distances from the edge of the plate, the deflections associated with body waves in the half-space are proportional to $x^{-3/2}$. Near the edge of the plate, two interface waves whose amplitudes decrease exponentially with distance from the edge may contribute significantly. The rate of decay of these interface waves decreases with decreasing frequency.

INTRODUCTION

In most, if not all, studies on the diffraction of elastic waves by a plane semi-infinite surface, the diffracting half-plane is either perfectly rigid (rigid barrier), rigid smooth or perfectly weak (semi-infinite open crack). In contrast to studies of that type, the present paper treats a problem in which the diffracting half-plane is itself an elastic body. The resulting scattered waves include the effects of dynamic interaction between the diffracting surface and the elastic continuum.

In this paper the diffracting surface is a thin, semi-infinite elastic plate in smooth contact with a homogeneous, isotropic, elastic half-space. A plane dilatational wave with harmonic time dependence is incident on the surface of the elastic solid, and the diffraction of the plane wave by the semi-infinite plate is studied. It is assumed that the frequency of the incident wave is sufficiently small, so that the Bernoulli–Euler theory can be used to describe the motion of the plate. By means of the bilateral Laplace transform and the Wiener–Hopf technique [1] an analytical expression is derived for the deflection of the plate. It is shown that dynamic interaction occurs primarily in the form of interface waves, where the interface waves correspond to certain roots of a characteristic equation, which is analyzed by a method outlined by Cagniard [2]. Similar interface waves were considered by Lamb [3], who treated the diffraction of a plane time-harmonic wave by a semi-infinite elastic plate immersed in an acoustic medium. By an improper choice of branch cuts, however, Lamb overlooked one root of the characteristic equation.

A limited number of steady-state time-harmonic elastodynamic half-plane diffraction problems have been discussed in the literature. Maue [4] discussed the diffraction of a plane time-harmonic wave by a perfectly weak half-plane (an open crack) in an infinite,

* Presently with the Division of Engineering, Brown University, Providence, Rhode Island.

homogeneous elastic solid. The propagation of a plane harmonic wave in an infinite elastic solid clamped along a half-plane (perfectly rigid half-plane) was considered by Roseau [5]. Thau and Pao [6] treated the diffraction of a plane harmonic wave by a rigid smooth half-plane. The diffraction of horizontal shear waves by a parabolic cylinder was also considered by Thau and Pao [7]. The latter problem reduces to a half-plane diffraction problem as a special case. Several transient diffraction problems, related to the above and to the problem considered here, are those discussed by de Hoop [8], Fredricks [9], and Freund and Achenbach [10,11]. An extensive list of references dealing with acoustic and electromagnetic half-plane diffraction problems is given by Noble [1].

FORMULATION

Let x, y, z be a three-dimensional cartesian coordinate system, oriented as in Fig. 1.

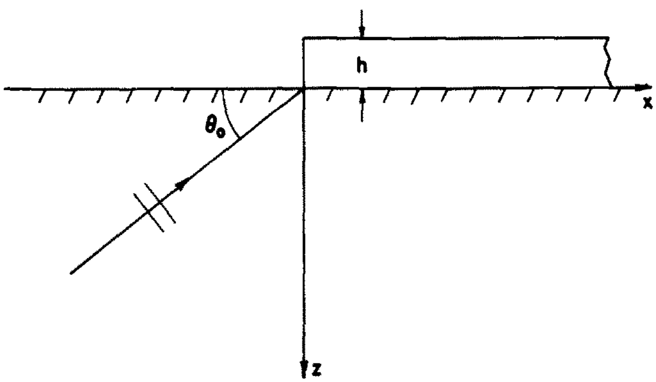

FIG. 1. Semi-infinite plate in smooth contact with elastic half-space.

The elastic solid occupies the region $z > 0$ and the y-axis coincides with the edge of the plate. The solid and the plate are in a state of plane strain, all field variables being independent of y. Since the incident disturbance is harmonic in time, with given frequency ω, we seek a steady-state response which is harmonic in time, with frequency ω. The common factor $e^{i\omega t}$, which appears in all field variables, is suppressed in the sequel. Wherever appropriate, the time derivative has been replaced by $i\omega$.

In the usual manner the displacement vector is expressed in terms of displacement potentials as

$$(u, v, w) = \left(\frac{\partial\varphi}{\partial x}-\frac{\partial\psi}{\partial z}, 0, \frac{\partial\varphi}{\partial z}+\frac{\partial\psi}{\partial x}\right). \tag{1}$$

The displacement potentials satisfy the reduced wave equations

$$\nabla^2\varphi+\varkappa_a^2\varphi = 0, \tag{2a}$$

$$\nabla^2\psi+\varkappa_b^2\psi = 0, \tag{2b}$$

where $\nabla^2 = \partial^2/\partial x^2+\partial^2/\partial z^2$. In (2a, b), $\varkappa_a = \omega a$ and $\varkappa_b = \omega b$, where a and b are the slownesses (i.e., inverse wave velocities) of dilatational and shear waves, respectively.

For a thin elastic plate, the equation governing transverse harmonic waves, whose wave length is large compared to the plate thickness, is

$$D\frac{\partial^4 w_p(x)}{\partial x^4} - m\omega^2 w_p(x) + p_z(x) = 0, \tag{3}$$

where $w_p(x)$ is the transverse displacement of the midplane and $p_z(x)$ is the transverse loading function. The bending stiffness D and the mass per unit surface area m are given by

$$D = E_p h^3/12(1-\nu_p^2), \qquad m = \rho_p h. \tag{4a, b}$$

In equations (4a, b), E_p, h, ν_p and ρ_p are, respectively, Young's modulus, the thickness, Poisson's ratio, and the mass density of the plate. The slowness c of time-harmonic transverse waves in a free plate is related to the frequency by

$$c^4 = 12\rho_p(1-\nu_p^2)/E_p h^2\omega^2, \tag{4c}$$

and we define the wave number $\varkappa_c = \omega c$.

The relevant stresses in the elastic solid are expressed, in terms of the displacement potentials, by

$$\sigma_{zz} = \mu\left(2\frac{\partial^2\psi}{\partial x\partial z} - 2\frac{\partial^2\varphi}{\partial x^2} - \varkappa_b^2\varphi\right), \tag{5a}$$

$$\sigma_{xz} = \mu\left(2\frac{\partial^2\varphi}{\partial x\partial z} + 2\frac{\partial^2\psi}{\partial x^2} + \varkappa_b^2\psi\right), \tag{5b}$$

where μ is the shear modulus.

The conditions to be satisfied along the boundary $z = 0$ are:

$$-\infty < x < \infty, \qquad \sigma_{xz}(x,0) = 0, \tag{6a}$$

$$-\infty < x < 0, \qquad \sigma_{zz}(x,0) = 0, \tag{6b}$$

$$0 < x < \infty, \qquad \sigma_{zz}(x,0) + p_z(x) = 0, \tag{6c}$$

$$0 < x < \infty, \qquad w(x,0) - w_p(x) = 0. \tag{6d}$$

We consider a plane dilatational wave, incident on the surface of the half-space,

$$\varphi_i(x,z) = \Phi_i e^{-i(\varkappa_0 x - \alpha_0 z)}. \tag{7}$$

The direction numbers of the plane wave normal are related to the angle of incidence θ_0 by $\varkappa_0 = \varkappa_a\cos\theta_0, \alpha_0 = \varkappa_a\sin\theta_0$.

When a wave of the form (7) is incident on the free surface of a semi-infinite elastic solid, it gives rise to a geometrically reflected dilatational wave

$$\varphi_r(x,z) = Ae^{-i(\varkappa_0 x + \alpha_0 z)}, \tag{8a}$$

and a reflected shear wave

$$\psi_r(x,z) = Be^{-i(\varkappa_0 x + \beta_0 z)}, \tag{8b}$$

where β_0 is defined by $\varkappa_0^2+\beta_0^2 = \varkappa_b^2$. The coefficients A and B, which are determined to satisfy the conditions of zero traction on the boundary $z = 0$, are

$$A = \frac{4\varkappa_0^2\alpha_0\beta_0-(\varkappa_b^2-2\varkappa_0^2)^2}{4\varkappa_0^2\alpha_0\beta_0+(\varkappa_b^2-2\varkappa_0^2)^2}\Phi_i, \tag{9a}$$

$$B = \frac{-4\varkappa_0\alpha_0(\varkappa_b^2-2\varkappa_0^2)}{4\varkappa_0^2\alpha_0\beta_0+(\varkappa_b^2-2\varkappa_0^2)^2}\Phi_i. \tag{9b}$$

To solve the posed diffraction problem we define "scattered" potentials φ_s and ψ_s by

$$\varphi = \varphi_i+\varphi_r+\varphi_s, \tag{10a}$$

$$\psi = \psi_r+\psi_s. \tag{10b}$$

The left sides of (10a, b) are the potentials introduced in (1) and represent all motion in the solid. The incident and reflected waves, given by (7) and (8), are assumed to be present for all x. The scattered potentials φ_s and ψ_s thus represent the effect of the plate on the system of plane waves traveling in the positive x-direction.

The scattered potentials must satisfy the reduced wave equations (2a, b) and the boundary conditions (6a, b). Substitution of (10a, b) into the boundary conditions (6c, d) yields two additional conditions on the scattered potentials in terms of the deflection and the loading function of the plate and the known reflected potentials

$$0 < x < \infty, \qquad \sigma_{zzs}(x, 0)+p_z(x) = 0, \tag{11a}$$

$$0 < x < \infty, \qquad w_s(x, 0)-iM(\varkappa_0)e^{-i\varkappa_0 x}-w_p(x) = 0, \tag{11b}$$

where

$$M(\varkappa_0) = (A-\Phi_i)\alpha_0+B\varkappa_0. \tag{12}$$

As pointed out in [10, 12], the conditions (6a, b) and (11a, b) are not sufficient to obtain a unique solution of the equations (2a, b). A unique solution is ensured, however, by imposing the additional condition that all displacements are continuous functions of x at $x = 0$, $z = 0$.

APPLICATION OF THE LAPLACE TRANSFORM

The diffraction problem can be solved by the Wiener–Hopf technique if there exists a strip in the complex plane of the transform parameter in which all transforms exist. Such a region of analyticity is obtained by introducing a slight spatial attenuation into the solid and the plate. This is accomplished by redefining the wave numbers $\varkappa_a$, $\varkappa_b$, $\varkappa_c$ so that each contains a small negative imaginary part. Thus,

$$\varkappa_a = \omega a(1-i\epsilon), \qquad \varkappa_b = \omega b(1-i\epsilon), \qquad \varkappa_c = \omega c(1-i\epsilon), \tag{13}$$

where $\epsilon > 0$ and, for convenience, all wave numbers have the same argument.

We introduce the bilateral Laplace transform on the spatial variable x, and we denote it by a bar over a function. Thus, $\bar{\varphi}_s$ is the transform of φ_s, and is defined by

$$\bar{\varphi}_s(\lambda, z) = \int_{-\infty}^{\infty} \varphi_s(x, z)e^{-\lambda x}\,dx. \tag{14}$$

The interval of convergence of this transform is determined by the asymptotic behavior of φ_s as $x \to \pm\infty$. As $x \to +\infty$, the dominant part of the scattered potential is the plane wave reflected from the plate which travels with an apparent surface wave number $\varkappa_0$. Thus, we write

$$\varphi_s(x, z) = O(\mathrm{e}^{-i\varkappa_0 x}) \quad \text{as } x \to +\infty, \qquad z \geq 0. \tag{15}$$

As $x \to -\infty$, the dominant part of φ_s is the cylindrical scattered wave, diverging from the virtual source at the origin, and we have

$$\varphi_s(x, z) = O(\mathrm{e}^{+i\varkappa_a x}) \quad \text{as } x \to -\infty, \qquad z \geq 0. \tag{16}$$

From conditions (15) and (16) and the fact that $\epsilon > 0$, we conclude that $\bar{\varphi}_s(\lambda, z)$ is an analytic function of λ in the strip Λ: $-\epsilon\omega a \cos\theta_0 < \operatorname{Re}\lambda < \epsilon\omega a$ of the complex λ-plane. Similar considerations show that $\bar{\psi}_s$ is also analytic in Λ.

By applying the transform (14) to the reduced wave equation for φ_s, we obtain an ordinary differential equation whose bounded solution is

$$\bar{\varphi}_s(\lambda, z) = P(\lambda)\mathrm{e}^{-i\alpha z}, \qquad \operatorname{Im}\alpha \leq 0, \tag{17}$$

where $P(\lambda)$ is an unknown function, analytic in Λ. The function α is defined by

$$\alpha = (\lambda^2 + \varkappa_a^2)^{\frac{1}{2}}. \tag{18}$$

The condition $\operatorname{Im}\alpha \leq 0$ is satisfied by providing branch cuts along curves for which $\operatorname{Im}\alpha = 0$, that is, where $(\lambda^2 + \varkappa_a^2)$ is real and positive. The branch cuts are defined by the segments of the hyperbola

$$(\operatorname{Re}\lambda)(\operatorname{Im}\lambda) = \epsilon\omega^2 a^2, \tag{19a}$$

$$(\operatorname{Re}\lambda)^2 - (\operatorname{Im}\lambda)^2 + \omega^2 a^2(1-\epsilon^2) > 0. \tag{19b}$$

The branch of α is chosen so that $\alpha = \varkappa_a$ when $\lambda = 0$. Similarly, we obtain

$$\bar{\psi}_s(\lambda, z) = Q(\lambda)\mathrm{e}^{-i\beta z}, \qquad \operatorname{Im}\beta \leq 0, \tag{20}$$

where $Q(\lambda)$ is an unknown function, analytic in Λ, and

$$\beta = (\lambda^2 + \varkappa_b^2)^{\frac{1}{2}}, \tag{21}$$

with appropriate branch cuts provided.

The bilateral transform is now applied to the boundary conditions. Condition (6a) becomes

$$\bar{\sigma}_{xzs}(x, 0) = 0. \tag{22a}$$

By employing equations (5b), (17), (20) and (22a), we obtain

$$Q(\lambda) = 2i\lambda\alpha P_1(\lambda), \tag{22b}$$

where $P_1(\lambda)$ is defined by

$$(2\lambda^2 + \varkappa_b^2)P_1(\lambda) = P(\lambda). \tag{23}$$

Condition (6b) reduces the bilateral transform of $\sigma_{zzs}(x, 0)$ to the one-sided transform, and we obtain

$$\bar{\sigma}_{zzs}(\lambda, 0) = -\mu H_+(\lambda), \tag{24a}$$

where $H_+(\lambda)$ is a suitable but undetermined function, analytic in the half-plane $\operatorname{Re}\lambda > -\epsilon\omega a\cos\theta_0$. With the aid of (22b), equation (24a) may be reduced to

$$R(\lambda)P_1(\lambda) = H_+(\lambda), \tag{24b}$$

where $R(\lambda)$ is the characteristic function of free surface Rayleigh waves,

$$R(\lambda) = (2\lambda^2+\varkappa_b^2)^2-4\lambda^2\alpha\beta. \tag{25}$$

To transform the remaining boundary conditions, we extend the definitions of $w_p(x)$ and $p_z(x)$ so that these functions satisfy (3) for $x > 0$ and $w_p(x) \equiv 0$, $p_z(x) \equiv 0$ for $x < 0$. Then, in view of (6b), the transform of condition (11a) is

$$\bar{\sigma}_{zz}(\lambda,0)+\bar{p}_z(\lambda) = 0, \tag{26}$$

which, upon use of (24a, b) and (3), may be reduced to

$$-\frac{\mu}{D}R(\lambda)P_1(\lambda)+g(\lambda)-(\lambda^4-\varkappa_c^4)F_+(\lambda) = 0, \tag{27}$$

where

$$g(\lambda) = w_p'''(0^+)+\lambda w_p''(0^+)+\lambda^2 w_p'(0^+)+\lambda^3 w_p(0^+) \tag{28}$$

and

$$F_+(\lambda) = \int_0^\infty w_p(x)e^{-\lambda x}\,dx. \tag{29}$$

The function $F_+(\lambda)$ is analytic in some right half-plane. Since we have introduced dissipation into the plate, the axis of convergence of the integral in (29) must be to the left of the axis $\operatorname{Re}\lambda = 0$.

The transform of the last boundary condition, (11b), is

$$\bar{w}_s(\lambda,0)-iM(\varkappa_0)(\lambda+i\varkappa_0)^{-1}-F_+(\lambda) = J_-^*(\lambda), \tag{30}$$

where $J_-^*(\lambda)$ is an undetermined function, analytic in the half-plane $\operatorname{Re}\lambda < \epsilon\omega a$. For convenience we now define a new function $J_-(\lambda)$ by

$$iM(\varkappa_0)(\lambda+i\varkappa_0)^{-1}+J_-^*(\lambda) = \frac{iM(\varkappa_0)J_-(\lambda)}{(\lambda+i\varkappa_0)J_-(-i\varkappa_0)}. \tag{31}$$

Equation (30) may then be reduced to

$$i\alpha\varkappa_b^2P_1(\lambda)+F_+(\lambda) = -\frac{iM(\varkappa_0)J_-(\lambda)}{(\lambda+i\varkappa_0)J_-(-i\varkappa_0)}. \tag{32}$$

The uniqueness condition of continuity of surface displacements at the origin and the standard Tauberian theorem for one-sided Laplace transforms [13] lead to the order conditions

$$H_+(\lambda),\ J_-(\lambda) = O(1) \quad \text{as } |\lambda| \to \infty. \tag{33}$$

SOLUTION BY MEANS OF THE WIENER–HOPF TECHNIQUE

Eliminating $F_+(\lambda)$ from equations (27) and (32) by subtraction, and employing (24b), we obtain the relation

$$\frac{\mu}{D}\alpha K(\lambda)H_+(\lambda) = g(\lambda)+iM(\varkappa_0)\frac{N(\lambda)J_-(\lambda)}{(\lambda+i\varkappa_0)J_-(-i\varkappa_0)}, \tag{34}$$

valid in Λ. In equation (34),

$$N(\lambda) = (\lambda^4-\varkappa_c^4), \tag{35a}$$

$$K(\lambda) = \frac{1}{\alpha}-ikh\frac{N(\lambda)}{R(\lambda)}, \tag{35b}$$

$$k = \varkappa_b^4\rho_p/\varkappa_c^4\rho, \tag{35c}$$

where ρ is the mass density of the elastic solid. The function $K(\lambda)$ is the characteristic function of waves at the interface of a plate and a semi-infinite elastic solid. The zeros of this characteristic function are discussed in Appendix 1. In Appendix 2 it is shown that $K(\lambda)$ may be factored into the product $K_+(\lambda)K_-(\lambda)$ where the factors have the following properties:

(i) $K_+(\lambda)$ is analytic for $\operatorname{Re}\lambda > -\epsilon\omega a$,
(ii) $K_-(\lambda)$ is analytic for $\operatorname{Re}\lambda < \epsilon\omega a$,
(iii) $K_\pm(\lambda) = O(\lambda)$ as $|\lambda| \to \infty$,
(iv) $K_+(-\lambda) = K_-(\lambda)$.

Once this factorization has been performed, equation (34) may be rewritten

$$\frac{\mu}{D}\frac{H_+(\lambda)K_+(\lambda)\alpha_+}{N_+(\lambda)} = \frac{g(\lambda)}{\alpha_-K_-(\lambda)N_+(\lambda)}+\frac{iM(\varkappa_0)N_-(\lambda)J_-(\lambda)}{\alpha_-K_-(\lambda)(\lambda+i\varkappa_0)J_-(-i\varkappa_0)}, \tag{36}$$

where

$$\alpha_\pm = (\varkappa_a\mp i\lambda)^{\frac{1}{2}}, \tag{37a}$$

$$N_\pm(\lambda) = (i\varkappa_c\pm\lambda)(\varkappa_c\pm\lambda). \tag{37b}$$

A further decomposition is required before the solution may be obtained. Let

$$L(\lambda) = \frac{g(\lambda)}{\alpha_-K_-(\lambda)N_+(\lambda)}, \tag{38}$$

and decompose $L(\lambda)$ into the sum $L_+(\lambda)+L_-(\lambda)$, where $L_+(\lambda)$ and $L_-(\lambda)$ are analytic in the half-planes $\operatorname{Re}\lambda > -\epsilon\omega a$ and $\operatorname{Re}\lambda < \epsilon\omega a$, respectively. Since the polynomial $g(\lambda)$ is an entire function, the only singularities in the left half-plane are simple poles at the zeros of $N_+(\lambda)$, and $L_+(\lambda)$ may be determined by inspection as

$$L_+(\lambda) = \frac{g(-i\varkappa_c)}{\varkappa_c(1-i)K_+(i\varkappa_c)\alpha_+(i\varkappa_c)(\lambda+i\varkappa_c)}-\frac{g(-\varkappa_c)}{\varkappa_c(1-i)K_+(\varkappa_c)\alpha_+(\varkappa_c)(\lambda+\varkappa_c)}. \tag{39a}$$

The function $L_-(\lambda)$ is then given by

$$L_-(\lambda) = L(\lambda)-L_+(\lambda). \tag{39b}$$

By employing equations (38) and (39b) we rewrite equation (36) in the form

$$\frac{\mu}{D}\frac{H_+(\lambda)K_+(\lambda)\alpha_+(\lambda+i\varkappa_0)}{N_+(\lambda)}-L_+(\lambda)(\lambda+i\varkappa_0)$$
$$=L_-(\lambda)(\lambda+i\varkappa_0)+iM(\varkappa_0)\frac{N_-(\lambda)J_-(\lambda)}{\alpha_-K_-(\lambda)J_-(-i\varkappa_0)}. \tag{40}$$

The left side of equation (40) is analytic in the half-plane Re $\lambda > -\epsilon\omega a\cos\theta_0$, the right side is analytic in Re $\lambda < \epsilon\omega a$, and the two sides are equal in Λ. Thus, each side is the analytic continuation of the other, and equals one and the same entire function, say $E(\lambda)$. By the order conditions (33), $E(\lambda) = O(\lambda^{\frac{1}{2}})$ as $|\lambda| \to \infty$ and, by an extension of Liouville's theorem, $E(\lambda)$ is a constant, say E_0. The value of E_0 is determined by setting $\lambda = -i\varkappa_0$ on the right side of (40),

$$E_0 = iM(\varkappa_0)\frac{N_+(i\varkappa_0)}{K_+(i\varkappa_0)\alpha_+(i\varkappa_0)}. \tag{41}$$

The function $H_+(\lambda)$ is determined from the left side of (40) as

$$\frac{\mu}{D}H_+(\lambda) = \frac{N_+(\lambda)}{\alpha_+K_+(\lambda)(\lambda+i\varkappa_0)}[E_0+L_+(\lambda)(\lambda+i\varkappa_0)]. \tag{42}$$

The transformed plate displacement is obtained from (24b) and (27) as

$$F_+(\lambda) = \frac{1}{N(\lambda)}\left[-\frac{\mu}{D}H_+(\lambda)+g(\lambda)\right]. \tag{43}$$

The term in the denominator of $F_+(\lambda)$ indicates the presence of poles at $\lambda = \varkappa_c,\ i\varkappa_c$. As pointed out after equation (29), however, $F_+(\lambda)$ must be regular at these points. Thus, we require

$$-\frac{\mu}{D}H_+(i\varkappa_c)+g(i\varkappa_c) = 0, \tag{44a}$$

$$-\frac{\mu}{D}H_+(\varkappa_c)+g(\varkappa_c) = 0. \tag{44b}$$

Two of the four constants appearing in $g(\lambda)$ are prescribed as plate boundary conditions. For example, if the edge of the plate is free we have $w_p''(0^+) = w_p'''(0^+) = 0$; if the edge is pinned we have $w_p(0^+) = w_p''(0^+) = 0$; etc. The conditions (44) are sufficient to determine the two constants in $g(\lambda)$ which are not prescribed as forced boundary conditions.

The motion of the plate is given by the inverse transform

$$w_p(x) = \frac{1}{2\pi i}\int_{Br} F_+(\lambda)e^{\lambda x}\,d\lambda, \tag{45}$$

where the path Br lies in Λ. The inversion contour and the branch lines of the integrand are shown in Fig. 2.

PLATE MOTION

The deflection of the plate is determined by evaluating the integral in (45). For $x > 0$, the contour Br may be completed by a semi-circle at infinity in the left half-plane, and $w_p(x)$ is obtained as the sum of the residues of the poles inside the closed contour plus the contribution from the branch line integrals. It follows from Appendix 1 that the singularities of $F_+(\lambda)$ are simple poles at $\lambda = -\lambda_1, -\lambda_2, -\lambda_3, -i\varkappa_0, -i\varkappa_c, -\varkappa_c$, and branch points at $\lambda = -i\varkappa_a, -i\varkappa_b$. The plate deflection $w_p(x)$ can then be written in the form

$$w_p(x) = \Sigma\,\mathrm{Res}(\mathrm{poles}) + \frac{1}{2\pi i}\int_{-\Gamma_+} F_+(\lambda)e^{\lambda x}\,\mathrm{d}\lambda, \tag{46}$$

where the path $\Gamma_+ = \Gamma_{a+} + \Gamma_{b+}$ is shown in Fig. 2. The residues may be written as

$$\mathrm{Res}(-\lambda_j) = \frac{E_0 + L_+(-\lambda_j)(-\lambda_j + i\varkappa_0)}{N_+(\lambda_j)(-\lambda_j + i\varkappa_0)}\left[\frac{(\lambda + \lambda_j)}{\alpha_+ K_+(\lambda)}\right]_{\lambda = -\lambda_j} \mathrm{e}^{-\lambda_j x}, \qquad j = 1, 2, 3, \tag{47}$$

$$\mathrm{Res}(-\varkappa_c) = \mathrm{Res}(-i\varkappa_c) = 0, \tag{48}$$

$$\mathrm{Res}(-i\varkappa_0) = -\frac{iM(\varkappa_0)}{\alpha(i\varkappa_0)K(i\varkappa_0)}\mathrm{e}^{-i\varkappa_0 x}. \tag{49}$$

The complicated form of $F_+(\lambda)$ makes exact evaluation of the branch line integrals in (46) impossible. A first order approximation for large values of x, however, is quite easy to obtain by a method discussed in detail by Ewing *et al.*[14]. We present only the results here.

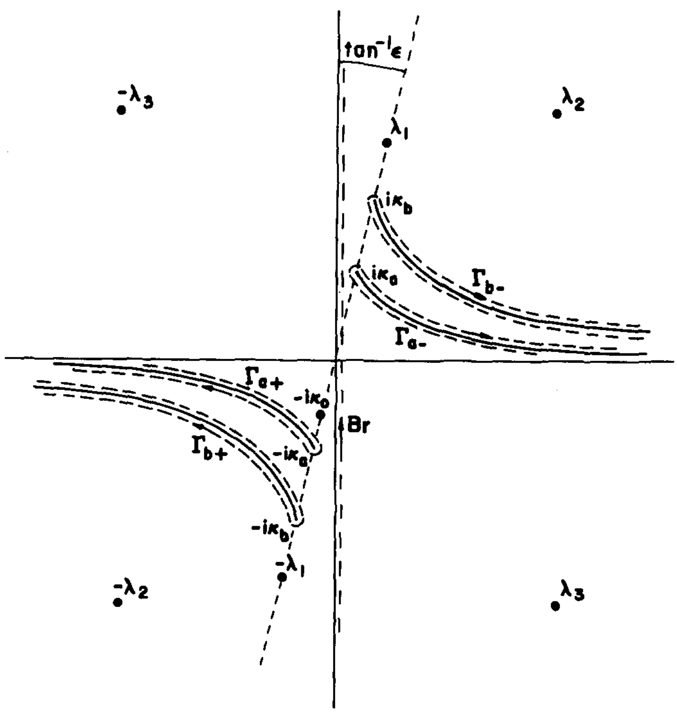

FIG. 2. Inversion contour in λ-plane.

The principal contributions to the integrals arise from integration near the branch points, and the asymptotic values of the integrals are

$$\frac{1}{2\pi i}\int_{-\Gamma_{a+}} F_+(\lambda)e^{\lambda x}\,d\lambda = A_a\left(\frac{\varkappa_a}{2\pi}\right)^{\frac{1}{2}} e^{i(\pi/4)}e^{-i\varkappa_a x}(x)^{-\frac{3}{2}}, \tag{50a}$$

$$\frac{1}{2\pi i}\int_{-\Gamma_{b+}} F_+(\lambda)e^{\lambda x}\,d\lambda = A_b\left(\frac{\varkappa_b}{2\pi}\right)^{\frac{1}{2}} e^{i(\pi/4)}e^{-i\varkappa_b x}(x)^{-\frac{3}{2}}, \tag{50b}$$

where

$$A_a = \frac{khN_-(i\varkappa_a)(2\varkappa_a)^{\frac{1}{2}}K_+(i\varkappa_a)}{(\varkappa_b^2-2\varkappa_a^2)^2(i\varkappa_0-i\varkappa_a)}[E_0+L_+(-i\varkappa_a)(i\varkappa_0-i\varkappa_a)], \tag{51a}$$

$$A_b = \frac{8D\varkappa_b^4 N_-(i\varkappa_b)\alpha_+(i\varkappa_b)K_+(i\varkappa_b)(\varkappa_b^2-\varkappa_a^2)}{\mu[\varkappa_b^4-kh(\varkappa_b^4-\varkappa_c^4)(\varkappa_b^2-\varkappa_a^2)^{\frac{1}{2}}]^2(i\varkappa_0-i\varkappa_b)}[E_0+L_+(-i\varkappa_b)(i\varkappa_0-i\varkappa_b)]. \tag{51b}$$

It might seem from (46) that, as $\epsilon \to 0$, we should obtain a contribution to the branch line integral near $\lambda = 0$. In fact, the horizontal and vertical legs of Γ_+ each contribute a term of order of x^{-1} as $x \to \infty$. The terms are, however, of equal magnitude and opposite sign and cancel each other.

If the edge of the plate is free we have $w_p''(0^+) = w_p'''(0^+) = 0$, and the deflection and the slope at $x = 0^+$ may be computed from equations (44a,b). We obtain

$$w_p(0^+) = -\frac{2(1+i)M(\varkappa_0)}{\varkappa_c J_+(i\varkappa_0)}\frac{(\varkappa_c+\varkappa_0)C_1/J_+(\varkappa_c)+(\varkappa_c+i\varkappa_0)C_3/J_+(i\varkappa_c)}{C_1C_4-iC_2C_3}, \tag{52}$$

where

$$J_+(\lambda) = K_+(\lambda)\alpha_+(\lambda) \tag{53}$$

$$C_1 = 1-i/[J_+(i\varkappa_c)]^2+(1-i)/J_+(i\varkappa_c)J_+(\varkappa_c) \tag{54a}$$

$$C_2 = 1+i/[J_+(i\varkappa_c)]^2+(1+i)/J_+(i\varkappa_c)J_+(\varkappa_c) \tag{54b}$$

$$C_3 = 1+(1+i)/J_+(\varkappa_c)J_+(i\varkappa_c)+i/[J_+(\varkappa_c)]^2 \tag{54c}$$

$$C_4 = 1+(1-i)/J_+(\varkappa_c)J_+(i\varkappa_c)-i/[J_+(\varkappa_c)]^2 \tag{54d}$$

In a similar manner we compute

$$w_p'(0^+) = \frac{2(1+i)M(\varkappa_0)}{J_+(i\varkappa_0)}\frac{(\varkappa_c+i\varkappa_0)C_4/J_+(i\varkappa_c)+i(\varkappa_c+\varkappa_0)C_2/J_+(\varkappa_c)}{C_1C_4-iC_2C_3}. \tag{55}$$

The solution of the posed diffraction problem is obtained by letting $\epsilon \to 0$ and multiplying all results by $e^{i\omega t}$.

DISCUSSION

In this paper we obtained analytical expressions for the various components of the deflection of the plate. Equations (50a,b) represent the motion of the plate associated with the dilatational and shear waves in the half-space. The body waves are cylindrical waves diverging from the virtual source at the edge of the plate. The amplitudes of these waves are proportional to $(x)^{-\frac{3}{2}}$, for large x.

The residue (49) represents the deflection of the plate due to the geometrical reflection of the incident wave from the interface for $x > 0$. The quantity $\mathrm{Res}(-i\varkappa_0)$ is, in fact, the amplitude of the plate deflection when a wave of the form (7) is incident on the interface of an infinite plate and an elastic half-space. This transverse wave in the plate travels unattenuated and with apparent surface wave number $\varkappa_0$.

Equation (48) implies that the amplitudes of terms corresponding to free vibrations of the free plate vanish identically.

The residue $\mathrm{Res}(-\lambda_1)$ represents a wave traveling unattenuated along the plate with slowness $|\lambda_1|$. In the solid, the amplitude of the motion decays exponentially as z increases. This wave corresponds to the free transverse wave discussed by Achenbach and Keshava[15] and is analogous to the familiar free surface Rayleigh wave.

Finally, we consider $\mathrm{Res}(-\lambda_2)$ and $\mathrm{Res}(-\bar{\lambda}_2)$ together. Setting $\lambda_2 = \xi + i\eta$, the plate motion associated with these residues may be written as

$$\mathrm{Res}(-\lambda_2)+\mathrm{Res}(-\bar{\lambda}_2) = \mathrm{e}^{-\xi x}\{C(-\lambda_2)\mathrm{e}^{-i\eta x}+C(-\bar{\lambda}_2)\mathrm{e}^{i\eta x}\}. \tag{56}$$

Equation (52) represents two waves traveling in opposite directions with identical speeds. Since ξ decreases with decreasing ω, see Fig. 4, the rate of decay decreases as the frequency decreases. This effect can be explained if it is realized that the edge effect is induced because, for example, the edge must be free of bending moment. A decreasing frequency implies an increasing wave length and a smaller bending moment in the plate, and thus a weaker edge effect. By investigating the motion associated with these waves in the half-space, it becomes plausible that these two waves combine to form a standing wave whose amplitude decays exponentially as the distance from the edge of the plate increases. Unfortunately, the complexity of the residues makes analytical verification of the existence of the standing waves impossible. The likelihood that (56) represents a standing wave is increased, however, by results from some related problems. It can be shown, for example, that for a time-harmonic line load applied to an infinite plate overlying an elastic half-space, the residues corresponding to (56) do indeed give rise to standing waves [16]. Similar standing waves, arising from the reflection of extensional waves from the free edge of a free semi-infinite elastic plate, were discussed by Gazis and Mindlin [17], and by Torvik [18].

From the discussion in Appendix 1 it can be concluded that, for an increase of the frequency ω beyond a certain value, the number of roots of $K(\lambda) = 0$ in the upper half of the λ-plane increases from three to four. The inequality to be satisfied for the existence of four roots is $\mathrm{Im}\, K(i\varkappa_b) < 0$, or

$$1/h(\varkappa_b^2-\varkappa_a^2)^{\frac{1}{2}}+\rho_p/\rho > k, \tag{57}$$

where k is defined by equation (35c). If four roots exist, the symmetry properties of $K(\lambda)$ no longer dictate that one root of $K(\lambda) = 0$ must be pure imaginary, and all roots may assume complex values. The motion associated with these complex roots, unlike the motion associated with $\mathrm{Res}(-\lambda_1)$, will decay exponentially with distance from the edge of the plate. The case of four roots, however, need not be considered in detail since the magnitude of the frequency required for the existence of four roots is beyond the range of frequencies for which the elementary plate theory, used to derive (3), is valid.

For $\varkappa \ll 1$, some numerical results may be obtained by employing the approximations worked out in Appendix 1 and Appendix 2. For example, using equations (79), (86), (62) and (63), we may write

$$J_+(\varkappa_c) = K_+(\varkappa_c)\alpha_+(\varkappa_c) \simeq \left[2\left(1-\frac{a^2}{b^2}\right)\frac{1}{\bar{\mu}}\right]^{\frac{1}{4}}\left(\frac{i}{1+q}\right)^{\frac{1}{2}}\frac{1}{(\varkappa_c h)^{\frac{1}{4}}}. \tag{58}$$

A similar expression is easily obtained for $J_+(i\varkappa_c)$. By substituting $J_+(\varkappa_c)$ and $J_+(i\varkappa_c)$ into equation (52) we conclude by comparison with equation (49) that the deflection of the free end is larger than, but of the same order of magnitude as, the deflection due to the geometrical reflection of the incident wave.

It is of particular interest to compute $\mathrm{Res}(-\lambda_1)$, which represents the wave traveling unattenuated along the plate. From equation (41) we conclude that E_0 is approximately of the form $S\varkappa_c^2(\varkappa_c h)^{\frac{1}{4}}M(\varkappa_0)$, where the constant S is of order unity. By substituting equation (58) and the analogous expression for $J_+(i\varkappa_c)$ into equation (39a), it is concluded that L_+ is approximately of the form $T\varkappa_c(\varkappa_c h)^{\frac{1}{4}}M(\varkappa_0)$, where T is of order unity. By employing equations (47), we find $\mathrm{Res}(-\lambda_1) = U(\varkappa_c h)^{\frac{1}{4}}M(\varkappa_0)\varkappa^3$, where U is of order unity. Comparing this result with equation (49), we conclude that the unattenuated surface wave is negligibly small as compared to the deflection of the plate due to the geometrical deflection of the incident wave. Thus for $\varkappa \ll 1$ the effect of the free edge is very localized.

In summary, we have obtained an analytical expression for the deflection of the plate. The deflections associated with an interface wave and with the geometrical reflection of the incident wave have constant amplitudes along the entire length of the plate. At large distances from the edge of the plate, the deflections associated with body waves in the half-space are proportional to $(x)^{-\frac{3}{2}}$. Near the edge of the plate, two interface waves whose amplitudes decrease exponentially with distance from the edge of the plate contribute significantly to the total motion. The rate of decay of these interface waves decreases with decreasing frequency. Some quantitative results are presented for the case $\varkappa = \omega h(\rho/\mu)^{\frac{1}{2}} \ll 1$.

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APPENDIX 1

Roots of $K(\lambda) = 0$

The roots of $K(\lambda) = 0$ are analyzed by a method discussed by Cagniard [2]. Suppose we have a closed contour γ in the λ-plane such that $K(\lambda)$ is analytic inside and on γ, except at a finite number of poles. From residue theory we have the formula

$$\frac{1}{2\pi i}\int_{\gamma}\frac{K'(\lambda)}{K(\lambda)}\,\mathrm{d}\lambda = n-m, \tag{59}$$

where n is the number of zeros and m is the number of poles of $K(\lambda)$ interior to γ, with due regard to multiplicity. We now consider the transformation $\lambda \to K(\lambda)$, for which the integral becomes

$$\frac{1}{2\pi i}\int_{\Gamma}\frac{\mathrm{d}K}{K} = n-m, \tag{60}$$

where Γ is the image of γ in the K-plane. The integrand in equation (60) has a simple pole at $K = 0$, and $n-m$ is, therefore, the total number of counterclockwise turns Γ makes around $K = 0$ as λ moves once around γ in a counterclockwise direction.

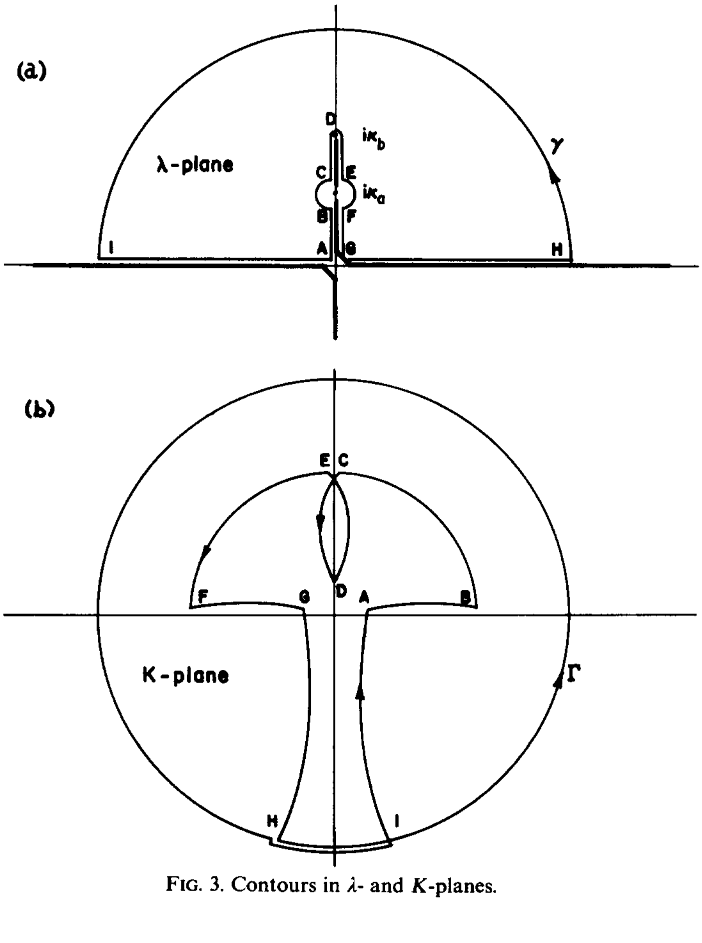

FIG. 3. Contours in λ- and K-planes.

Consider the contour γ shown in Fig. 3(a). Since $K(\lambda)$ is even, it suffices to consider the upper half-plane. The function $K(\lambda)$ has one simple pole interior to γ at the zero of $R(\lambda)$, and, therefore, $m = 1$. Under the condition $\operatorname{Im} K(i\varkappa_b) > 0$, the image path Γ is qualitatively shown in Fig. 3(b). It is noted that Γ encircles the point $K = 0$ twice, and thus $n = 3$, and $K(\lambda) = 0$ has three roots interior to γ, say at $\lambda = \lambda_1, \lambda_2, -\lambda_3$. Since $K(\lambda)$ is even, one of the roots, say λ_1, is pure imaginary when $\epsilon \to 0$; we also note $|\lambda_1| > \varkappa_b$. Furthermore, since $K(\bar{\lambda}) = -\bar{K}(\lambda)$, we have $\lambda_3 = \bar{\lambda}_2$. The root λ_1 corresponds to free interface waves and is discussed in [15]. The complex roots are relevant to forced motion problems. Some numerical values of λ_2 are given in Fig. 4, where

$$\varkappa = \varkappa_b h, \qquad \bar{\mu} = E_p/12\mu(1-\nu_p^2). \tag{61a,b}$$

For $\varkappa \ll 1$, it can be shown from (35b) that

$$\lambda_2 \simeq \left\{\frac{2\mu(1-a^2/b^2)}{D}\right\}^{\frac{1}{3}} e^{i(\pi/6)}. \tag{62}$$

For $\varkappa \ll 1$, the pure imaginary root may be written as

$$\lambda_1 \simeq i\varkappa_r(1-C_r\varkappa^3), \tag{63}$$

where

$$C_r = \bar{\mu}(r/b)(1-c^4/r^4)(1-a^2/r^2)^{\frac{1}{2}}/\bar{R}, \tag{64}$$

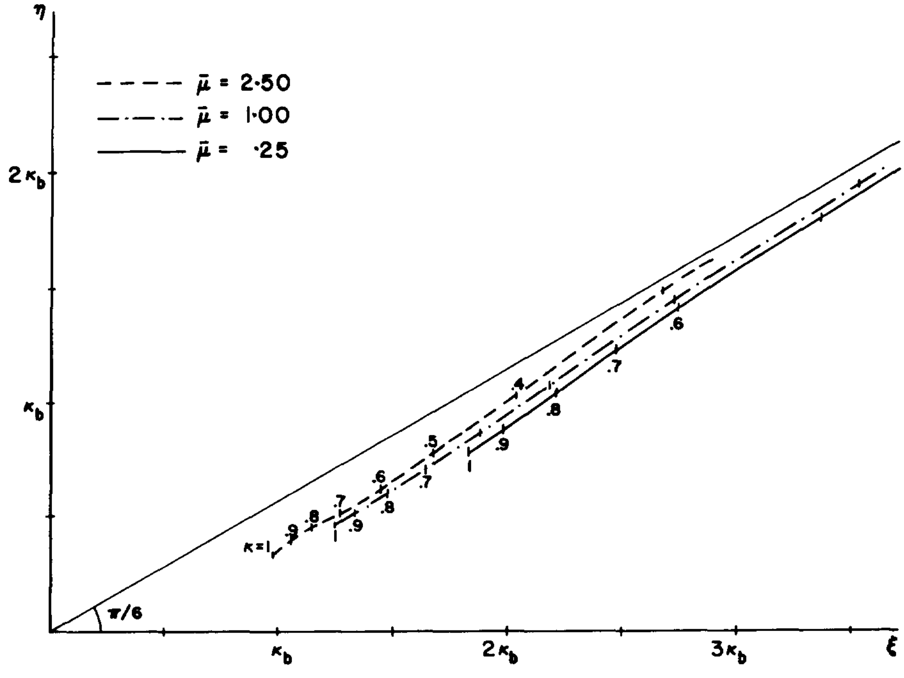

FIG. 4. Real and imaginary parts of $\lambda_2 = \xi + i\eta$ for various values of $\varkappa$ and $\bar{\mu}$.

wherein

$$r = \varkappa_r/\omega \tag{65}$$

$$\bar{R} = 8(b^2/r^2-2)+4(1-a^2/r^2)^{\frac{1}{2}}/(1-b^2/r^2)^{\frac{1}{2}} + 4(1-b^2/r^2)^{\frac{1}{2}}/(1-a^2/r^2)^{\frac{1}{2}} - 8(1-a^2/r^2)^{\frac{1}{2}}(1-b^2/r^2)^{\frac{1}{2}}. \tag{66}$$

APPENDIX 2

Factorization of $K(\lambda)$

To factor $K(\lambda)$, defined in (35b), such that the conditions (i)–(iv) are satisfied, we define a new function $K^*(\lambda)$ by

$$K^*(\lambda) = K(\lambda)\frac{(\lambda^2+\varkappa_r^2)(\lambda^2+\varkappa_a^2)}{k_1(\lambda_1^2-\lambda^2)(\lambda_2^2-\lambda^2)(\lambda_3^2-\lambda^2)}, \tag{67}$$

where

$$k_1 = ikh/2(\varkappa_b^2-\varkappa_a^2), \tag{68}$$

and $\lambda = \pm i\varkappa_r$ are the only roots of $R(\lambda) = 0$. The only singularities of $K^*(\lambda)$ are branch points at $\lambda = \pm i\varkappa_a$, $\pm i\varkappa_b$. Branch cuts are provided, consistent with the conditions that $\mathrm{Im}(\alpha, \beta) \le 0$. The function $K^*(\lambda)$ is nowhere real and negative, and $K^*(\lambda) \to 1$ as $|\lambda| \to \infty$.

We obtain the factors of $K(\lambda)$ by factoring $K^*(\lambda)$ into the product $K^*_+(\lambda)K^*_-(\lambda)$, where $K^*_+(\lambda)$ and $K^*_-(\lambda)$ are analytic and nonzero in the half-planes $\mathrm{Re}\,\lambda > -\epsilon\omega a$ and $\mathrm{Re}\,\lambda < \epsilon\omega a$, respectively, and $K^*_\pm(\lambda) \to 1$ as $|\lambda| \to \infty$. The factors are given by [10]

$$K^*_\mp(\lambda) = \exp\left\{\frac{1}{2\pi i}\int_{\Gamma_\pm} \ln K^*(\eta)\frac{\mathrm{d}\eta}{\eta-\lambda}\right\}, \tag{69}$$

where the contours $\Gamma_\pm = \Gamma_{a\pm}+\Gamma_{b\pm}$ are shown in Fig. 2. For $K^*_-(\lambda)$, subsequent simplification yields, for $\epsilon \to 0$,

$$K^*_-(\lambda) = \exp\left\{\frac{1}{2\pi}\int_{i\varkappa_a}^{i\varkappa_b} \chi_1(\eta)\frac{\mathrm{d}\eta}{\eta-\lambda}+\frac{1}{2\pi}\int_{i\varkappa_a}^{\infty} \chi_2(\eta)\frac{\mathrm{d}\eta}{\eta-\lambda}\right\}, \tag{70}$$

where

$$\chi_1(\eta) = 2\tan^{-1}\frac{|\alpha|^{-1}-khR_1^*}{khR_2^*}+3\pi \tag{71}$$

$$\chi_2(\eta) = -2\cot^{-1}(kh\alpha R^*), \tag{72}$$

wherein α is defined by equation (18), and

$$R^* = R_1^*+iR_2^* = N(\eta)/R(\eta). \tag{73}$$

The integrations are taken along the left and lower sides of the cut in the right half-plane, as $\epsilon \to 0$. The range of the arctangent is determined by the choice of branch of the logarithm as $-3\pi/2 \le \mathrm{Im}(\ln y) \le \pi/2$ for any y. The factor $K^*_+(\lambda)$ is equal to $K^*_-(-\lambda)$.

The factors of $K(\lambda)$ satisfying the required conditions (i)–(iv) are obtained from equation (67) as

$$K_{\pm}(\lambda) = K_{\pm}^{*}(\lambda)\frac{(k_1)^{\frac{1}{2}}(\lambda_1 \pm \lambda)(\lambda_2 \pm \lambda)(\lambda_3 \pm \lambda)}{(i\varkappa_r \pm \lambda)(i\varkappa_a \pm \lambda)}. \tag{74}$$

The integrals in equation (70) cannot be evaluated analytically without making certain approximations. Since the factor K_+ need be evaluated at a few specific points only, see equations (39a) and (47–51b), it would be feasible to evaluate the integrals by a numerical procedure. Considerable insight in the response of the system can, however, be gained by considering an approximate evaluation. For this purpose we note that for $\varkappa = \varkappa_b h \ll 1$:

$$\chi_1(\eta) = O(\varkappa^2) \qquad \varkappa_a \leq |\eta| \leq \varkappa_b \tag{75}$$

$$\chi_2(\eta) = -\pi + O(\varkappa^2) \qquad 0 \leq |\eta| \leq \varkappa_a. \tag{76}$$

In view of equation (75) the integral between the limits $i\varkappa_a$ and $i\varkappa_b$ may then be neglected. In view of equation (76), and since $\chi_2(\eta) = O(\eta^{-3})$ for $\eta \to \infty$, and with the purpose of obtaining a convenient expression for $K_+(\lambda)$, we now assume that we can find a point ζ on the real η-axis such that

$$\frac{1}{2\pi}\int_{i\varkappa_a}^{\infty} \chi_2(\eta)\frac{\mathrm{d}\eta}{\eta-\lambda} = \frac{1}{2\pi}\int_{i\varkappa_a}^{\zeta} (-\pi)\frac{\mathrm{d}\eta}{\eta-\lambda}. \tag{77}$$

Assuming that ζ can be found, the integral on the right-hand side of equation (71) can be evaluated, and we find for $\varkappa \ll 1$,

$$K_{-}^{*}(\lambda) \simeq (i\varkappa_a - \lambda)^{\frac{1}{2}}/(\zeta - \lambda)^{\frac{1}{2}}. \tag{78}$$

By employing the relation $K_{+}^{*}(\lambda) = K_{-}^{*}(-\lambda)$ and equation (74), we obtain

$$K_{+}(\lambda)\alpha_{+}(\lambda) = \frac{(k_1)^{\frac{1}{2}}(\lambda_1 + \lambda)(\lambda_2 + \lambda)(\lambda_3 + \lambda)}{(i\varkappa_r + \lambda)(\zeta + \lambda)^{\frac{1}{2}}}. \tag{79}$$

The function $\zeta(\lambda)$ still remains to be determined from the equation

$$\tfrac{1}{2}\ln(\zeta - \lambda) = -\frac{1}{2\pi}\int_{i\varkappa_a}^{\infty} \chi_2(\eta)\frac{\mathrm{d}\eta}{\eta - \lambda} + \tfrac{1}{2}\ln(i\varkappa_a - \lambda). \tag{80}$$

After integration by parts, and using equation (76), equation (80) may be reduced to

$$\tfrac{1}{2}\ln(\zeta - \lambda) = \frac{1}{\pi}\int_0^{\infty} \frac{\ln(\eta - \lambda)}{1 + (khR^*\alpha)^2}\frac{\mathrm{d}}{\mathrm{d}\eta}(khR^*\alpha)\,\mathrm{d}\eta. \tag{81}$$

It is noted from equation (81) that the integrand is of order $\varkappa^3$ for small η. We are therefore justified in using an approximation for the integrand valid for $\eta \gg \varkappa_a$, and we write

$$\tfrac{1}{2}\ln(\zeta - \lambda) = \frac{1}{\pi}\int_0^{\infty} \frac{\ln(\eta - \lambda)}{1 + (p\eta^3)^2}\frac{\mathrm{d}}{\mathrm{d}\eta}(p\eta^3)\,\mathrm{d}\eta \tag{82}$$

where

$$p = kh/2(\varkappa_b^2 - \varkappa_a^2). \tag{83}$$

Since the part of the integrand multiplying $\ln(\eta-\lambda)$ shows a maximum at $\bar{\eta} = (\frac{1}{2}p^2)^{\frac{1}{6}}$, the main contribution to the integral most likely comes from the vicinity of $\eta = \bar{\eta}$, and we approximate finally,

$$\tfrac{1}{2}\ln(\zeta-\lambda) \simeq \frac{1}{\pi}\ln(\bar{\eta}-\lambda)\int_0^\infty \frac{(\mathrm{d}/\mathrm{d}\eta)(p\eta^3)}{1+(p\eta^3)^2}\,\mathrm{d}\eta = \tfrac{1}{2}\ln(\bar{\eta}-\lambda). \tag{84}$$

From equation (84) we conclude

$$\zeta \simeq \bar{\eta} = q\varkappa_c, \tag{85}$$

where

$$q = \left[\left(1-\frac{a^2}{b^2}\right)^2 \frac{\rho}{\rho_p}\frac{\varkappa_c^2}{\varkappa_b^2}\frac{1}{\varkappa^2}\right]^{\frac{1}{6}}. \tag{86}$$

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Абстракт—Исследуется движение волны в полубесконечной упругой пластинке, находящейся в контакте без трения с упругим полупространством. Волны в пластинке, вызываные плоской, гармонической по времени волнои дилатации, падают на поверхность полупространства. Используя методы преобразования Лапласа и способ Винера-Хопфа определяется аналитическая формула для изгиба пластинки. Изгиб заключает члены, свзанные с пространственными волнами в полупространстве, с поверхностными волнами, принадлежащими к поверхности раздела между пластинкой и полупространством, а также с геометрическим Отряжением падающей волны. Изгибы, связанные с волной поверхности раздела и с геометрическим отряжением падающей волны имеют постоянные амплитуды по всей длине пластинки. При больших расстояниях от края, изгибы связанные с ппространственными волнами в полупространстве являются пропорциональными к $x^{-3/2}$. Близи края пластинки, две волны поверхности раздела, амплитуды которых уменьшаются экспотенциально с растоянием от края, могут иметь большое значение. Скорость затухания этих волн поверхности раздела уменьмается с частотой затухания.